A NOTE ON A SET OF FUNCTIONALS FOR SPECTRAL ANALYSES OF NON-STATIONARY PROCESSES ON A FINITE INTERVAL*

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1. Introduction

Spectral analyses of non-stationary random processes are needed when analyzing observations of random phenomena in a changing physical environment. A class of such processes, which are continuous with respect to a time parameter, is generated by associating a random component, such as noise, with linear or non-linear trends, periodic components or a combination of these.

A set of functionals and their statistical estimates are defined for these time dependent processes over subintervals contained in a finite interval of time representing the duration of observation. The mathematical expectation of processes formed from the set of functionals provide representations for the covariance and frequency spectrum which have statistically consistent estimators. These representations of the covariance and frequency spectrum become the well known definitions of the centralized autocovariance and power spectrum for wide-sense stationary processes when that assumption is invoked.

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2. Notation, Definitions, and Assumptions

We define a continuous parameter random process $\{Y_t; t\epsilon[T^*,T^{**}]\}$ to be strictly non-stationary over the finite interval of observation $[T^*,T^{**}]$ if all orders of the associated probability distributions are time dependent and consequently all orders of the defining moments are continuous functions of time. The process is said to be wide sense non-stationary if at least the probability distributions associated with the covariance function are time dependent. Functionals and relations between them that are valid when the associated distributions are time dependent, are also true for time independent distributions. Thus in all statements and proofs we assume time dependent distributions. Further, we assume that those orders of the process moments necessary for proofs exist for each $t\epsilon[T^*,T^{**}]$ and that each such moment is integrable in the Riemann sense over that interval.

Given a continuous parameter non-stationary random process $\{Y_t; t\epsilon[T^*,T^{**}]\}$ and sample functions $y(t)$ thereof, we form sequences of subintervals $[t'-S,t'+S]$ dependent on a midtime t' such that

i. Y_t and $y(t)$ are considered for $t'-T \le t \le t'+T, \quad 0 \le T \le S$

and

ii. $Y_{t+\tau}$ and $y(t+\tau)$ are considered for $t'-T \le t \le t' + T$, and $T-S \le \tau \le S-T$.

That is, for each $t' \epsilon [T^*,T^{**}]$, we require

$$[t'-T,t'+T] \subseteq [t'-S,t'+S] \subset [T^*,T^{**}]$$

We define the linear random functionals (stochastic integrals)

$$A_T(Y_t;t') = \frac{1}{2T}\int_{t'-T}^{t'+T} Y_t dt$$

$$A_T(Y_{t+\tau};t') = \frac{1}{2T}\int_{t'-T}^{t'+T} Y_{t+\tau} dt \quad ,$$

$$A_T(Y_t Y_{t+\tau};t') = \frac{1}{2T}\int_{t'-T}^{t'+T} Y_t Y_{t+\tau} dt \tag{2.1}$$

to exist for each real function Y_t of the process $\{Y_t; t\epsilon[T^*,T^{**}]\}$ where $\tau\epsilon[T-S,S-T]$. The process moments and their time integrals that we assume to exist are

$$-\infty < EY_t^n < \infty$$

$$-\infty < EY_{t+\tau}^n < \infty$$

$$-\infty < EY_t^n Y_{t+\tau}^n < \infty, \tag{2.2}$$

for n=1,2,3,4, and

$$-\infty < \int_{t'-T}^{t'+T} EY^n dt < \infty$$

$$-\infty < \int_{t'-T}^{t'+T} EY_{t+\tau}^n dt < \infty$$

$$-\infty < \int_{t'-T}^{t'T} EY_t^n Y_{t+\tau}^n dt < \infty \tag{2.3}$$

for $t\epsilon[t'-T,t'+T], \subset[t'-S,t'+S] \subset[T^*,T^{**}]$ and $\tau\epsilon[T-S,S-T]$. With the assumption that the linear operations of mathematical expectation and integration with respect to the parameter may be interchanged, the

following statements may be made for the functionals of (2.1)

$$EA_T(Y_t^n;t') = A_T(EY_t^n;t')$$

$$EA_T(Y_{t+\tau}^n;t') = A_T(EY_{t+\tau}^n;t')$$

$$EA_T(Y_t^nY_{t+\tau}^n;t') = A_T(EY_t^nY_{t+\tau}^n;t'). \qquad (2.4)$$

With these basic assumptions and definitions, we proceed to generalized definitions for the covariance, spectral representation, their estimators, and proofs for the consistency of these estimators.

3. Covariance

The centralized covariance for the non-stationary random process $\{Y_t;t\epsilon[T^*,T^{**}]\}$ is defined with the functionals of (2.1) by

$$\psi_T(\tau;t') = A_T(EY_tY_{t+\tau};t') - A_T(EY_t;t')A_T(EY_{t+\tau};t') \qquad (3.1)$$

for $\tau\epsilon[T-S,S-T]$ and any subinterval $[t'-T,t'+T] \subset [t'-S,t'+S] \subset [T^*,T^{**}]$. For wide sense stationary processes the definition of (3.1) reduces to

$$\psi_T(\tau) = EY_tY_{t+\tau} - E^2Y_t$$

and to

$$\psi_T(\tau;t') = A_T(Y_tY_{t+\tau};t') - A_T(Y_t;t')A_T(Y_{t+\tau};t')$$

for deterministic processes.

An important consideration concerning the properties of the covariance of a non-stationary process is that it need not be an even function of τ. This is evident when one considers a process involving a time dependent non-trivial polynomial trend. Therefore, unlike wide sense stationary processes, this property must be reflected in the definition of a spectral representation for non-stationary processes.

4. A Spectral Representation

Since it is possible that

$$\psi_T(\tau;t') \neq \psi(-\tau;t') \tag{4.1}$$

for $\tau\epsilon[T-S,S-T]$, then a spectral representation for a non-stationary process over $[t'-S,t'+S] \subset [T^*,T^{**}]$

$$\begin{aligned}\Phi(nw_{S-T};t') &= \left| \frac{1}{2(S-T)} \int_{T-S}^{S-T} \psi_T(\tau;t')\cos(nw_{S-T}\tau)d\tau \right| \\ &\qquad + \left| \frac{1}{2(S-T)} \int_{T-S}^{S-T} \psi_T(\tau;t')\sin(nw_{S-T}\tau)d\tau \right| \\ &= \left| A_{S-T}[\psi_T(\tau;t')\cos(nw_{S-T}\tau)] \right| \\ &\qquad + \left| A_{S-T}[\psi_T(\tau;t')\sin(nw_{S-T}\tau)] \right| \end{aligned} \tag{4.2}$$

where $w_{S-T} = \pi/(S-T)$. This spectral representation for a non-stationary process has the desired properties that

a. $\Phi(nw_{S-T};t') \geq 0$ for $nw_{S-T} \geq 0$,

b. if (4.1) holds, the Fourier sine transform is non-zero and is not neglected, and

c. for $\psi_T(\tau;t') = \psi_T(-\tau;t')$, the definition of $\Phi(nw_{S-T};t')$ is

$$\Phi(nw_{S-T};t') = A_{S-T}[\psi_T(\tau;t') \cos(nw_{S-T}\tau)]$$

which is the power spectrum often stated for random-periodic processes.

In general, the spectral representation (4.2) no longer enjoys the inverse transform relationship with $\psi_T(\tau;t')$. The inverse transform relation does occur for the preceding property c, i.e., when the Fourier sine transform is zero and $\Phi(nw_{S-T};t')$ is positive definite.

5. An Estimate for the Covariance Functional

Assume that we possess M independent sample functions $y_m(t)$, $m=1,2,\ldots,M$, of the continuous non-stationary random process $\{Y_t; t\in[T^*,T^{**}]\}$. Then for all t on any subinterval $[t'-T,t'+T] \subseteq [t'-S,t'+S] \subseteq [T^*,T^{**}]$ and $\tau\in[S-T,T-S]$, the following expectations exist

$$
\begin{aligned}
Ey_m(t) &= EY_t \\
Ey_m(t+\tau) &= EY_{t+\tau} \\
Ey_m(t)y_m(t+\tau) &= EY_t Y_{t+\tau}
\end{aligned}
\tag{5.1}
$$

for each $m=1,2,\ldots,M$. From (2.4) we have

$$
\begin{aligned}
EA_T[y_m(t);t'] &= A_T[Ey_m(t);t'] = A_T(EY_t;t') \\
EA_T[y_m(t+\tau);t'] &= A_T[Ey_m(t+\tau);t'] = A_T(EY_{t+\tau};t') \\
EA_T[y_m(t)y_m(t+\tau);t'] &= A_T[Ey_m(t)y_m(t+\tau);t'] = A_T(EY_t Y_{t+\tau};t').
\end{aligned}
\tag{5.2}
$$

Next, we define an estimate for the covariance functional of (3.1) in terms of the M sample functions $y_m(t)$ as

$$
\psi_{MT}(\tau;t') = \frac{1}{M}\sum_{i=1}^{M} A_T[y_i(t)y_i(t+\tau);t'] - \frac{1}{M^2}\sum_{j=1}^{M} A_T[y_j(t);t'] \sum_{k=1}^{M} A_T[y_k(t+\tau);t'] \tag{5.3}
$$

and observe that we can rewrite this estimate as

$$\psi_{MT}(\tau;t') = \frac{1}{M^3} \sum_{i,j,k=1}^{M} \{A_T[y_i(t)y_i(t+\tau);t'] - A_T[y_j(t);t']A_T[y_k(t+\tau);t']\}. \tag{5.4}$$

The consistency of this estimate is stated in the following theorem and then proved.

<u>Theorem</u>. If $|\psi_T(\tau;t')|^2 < \infty$ for all $t\epsilon[t'-T,t'+T] \subset [t'-S,t'+S] \subset [T^*,T^{**}]$ and all $\tau\epsilon[T-S,S-T]$, then $E|\psi_{MT}(\tau;t')|^2 < \infty$ and

$$\lim_{M\to\infty} E|\psi_{MT}(\tau;t') - \psi_T(\tau;t')|^2 = 0.$$

Thus $P \lim_{M\to\infty} \psi_{MT}(\tau;t') = \psi_T(\tau;t')$ and hence $\psi_{MT}(\tau;t')$ is a consistent estimate for $\psi_T(\tau;t')$.

<u>Proof</u>: Write

$$E|\psi_{MT}(\tau;t') - \psi_T(\tau;t')|^2 = E[\psi_{MT}(\tau;t') - \psi_T(\tau;t')]^2$$
$$= E\psi_{MT}^2(\tau;t') - 2\psi_T(\tau;t')E\psi_{MT}(\tau;t') + \psi_T^2(\tau;t'); \tag{5.5}$$

and let

$$a_i = A_T[y_i(t)y_i(t+\tau);t']$$
$$b_j = A_T[y_j(t);t']$$
$$c_k = A_T[y_k(t+\tau);t']. \tag{5.6}$$

Then the term $E\psi_{MT}(\tau;t')$ of (5.5) may be written as

$$E\psi_{MT}(\tau;t') = \frac{1}{M^3}\sum_{i,j,k=1}^{M} E(a_i - b_j c_k) = \frac{1}{M}\sum_{i=1}^{M} E(a_i)$$
$$- \frac{1}{M^2}\sum_{j=1}^{M} E(b_j c_j) - \frac{1}{M^2}\sum_{j,k=1}^{M} E(b_j)E(c_k)$$

where

$$E(b_j c_j) = E\{A_T[y_j(t);t']A_T[y_j(t+\tau);t']\}$$
$$\leq A_T[Ey_j^2(t);t'] + A_T[Ey_j^2(t+\tau);t'].$$

Then by (3.1) and (5.2),

$$E\psi_{MT}(\tau;t') \geq \psi_T(\tau;t') - \frac{1}{M}\left|A_T(EY_t^2 + EY_{t+\tau};t') - A_T(EY_t;t')A_T(EY_{t+\tau};t')\right|.$$

But by (2.3) there exists a positive number $k_1 < \infty$ such that

$$\left|A_T(EY_t^2 + EY_{t+\tau}^2;t') - A_T(EY_t;t')A_T(EY_{t+\tau};t')\right| \leq k_1$$

for all $t' \epsilon [T^*,T^{**}]$ and all $\tau \epsilon [T-S,S-T]$. Therefore

$$E\psi_{MT}(\tau;t') \geq \psi_T(\tau;t') - \frac{k_1}{M} \tag{5.7}$$

Similarly the term $E\psi_{MT}^2(\tau;t')$ of (5.5) may be calculated from

$$E\psi_{MT}^2(\tau;t') = E\{[\frac{1}{M^3}\sum_{i,j,k=1}^{M}(a_i - b_j c_k)][\frac{1}{M^3}\sum_{m,n,p=1}^{M}(a_m - b_n c_p)]\}$$
$$= \frac{1}{M^6}\sum_{i,j,k,m,n,p=1}^{M} E[(a_i - b_j c_k)(a_m - b_n c_p)]$$
$$= \frac{1}{M^2}\sum_{i,m=1}^{M} E(a_i a_m) - \frac{2}{M^3}\sum_{i,n,p=1}^{M} E(a_i b_n c_p) + \frac{1}{M^4}\sum_{j,k,n,p=1}^{M}(b_j c_k b_n c_p).$$

Then by (3.1) and (5.2)

$$E\psi^2_{MT}(\tau;t') \le \psi^2_T + \frac{K}{M}\,[A_T(EY^4_t + EY^4_{t+\tau};t')]$$

where $K = f(M^{-1}, M^{-2})$. Therefore, by (2.3), there exists a positive number $k_2 < \infty$ such that

$$K[A_T(EY^4_t + EY^4_{t+\tau};t')] \le k_2$$

and hence

$$E\psi^2_{MT}(\tau;t') \le \psi^2_T(\tau;t') + \frac{k_2}{M}. \tag{5.8}$$

Substituting (5.7) and (5.8) into (5.6)

$$E|\psi_{MT}(\tau;t') - \psi_T(\tau;t')|^2 \le \psi^2_T(\tau;t') + \frac{k_2}{M} - 2\psi^2_T(\tau;t') + \frac{2k_1}{M}\,\psi_T(\tau;t')$$

$$+ \psi^2_T(\tau;t') \le \frac{1}{M}\,[k_2 + 2k_1\psi_T(\tau;t')]$$

Since it was assumed that $|\psi_T(\tau;t')|^2 < \infty$, then

$$\lim_{M\to\infty} E|\psi_{MT}(\tau;t') - \psi_T(\tau;t')|^2 = 0$$

for all $t\epsilon[t'-T,t'+T] \subset [t'-S,t'+S] \subset [T^*,T^{**}]$ and all $\tau\epsilon[T-S,S-T]$.

6. An Estimate for the Spectral Representation

Let

$$\Phi_M(nw_{S-T};t') = |A_{S-T}[\psi_{MT}(\tau;t')\cos(nw_{S-T}\tau)]|$$

$$+ |A_{S-T}[\psi_{MT}(\tau;t')\sin(nw_{S-T}\tau)]|$$

be an estimate of the spectral representation $\Phi(nw_{S-T};t')$ as given by (4.2). For $\Phi_M(nw_{S-T};t')$ to be a consistent estimate of $\Phi(nw_{S-T};t')$,

we show that

$$\lim_{M\to\infty} E\left|\Phi_M(nw_{S-T};t') - \Phi(nw_{S-T};t')\right|^2 = 0$$

Consider

$$\begin{aligned}
\left|\Phi_M(nw_{S-T};t') - \Phi(nw_{S-T};t')\right| = &\Big|\left|A_{S-T}[\psi_{MT}(\tau;t')\cos(nw_{S-T}\tau)]\right| \\
&+ \left|A_{S-T}[\psi_{MT}(\tau;t')\sin(nw_{S-T}\tau)]\right| \\
&- \left|A_{S-T}[\psi_{T}(\tau;t')\cos(nw_{S-T}\tau)]\right| \\
&- \left|A_{S-T}[\psi_{T}(\tau;t')\sin(nw_{S-T}\tau)]\right|\Big| \\
\le &\left|A_{S-T}\{[\psi_{MT}(\tau;t') - \psi_T(\tau;t')\{\cos(nw_{S-T}\tau)\}\right| \\
&+ \left|A_{S-T}\{[\psi_{MT}(\tau;t') - \psi_T(\tau;t')]\sin(nw_{S-T}\tau)\}\right| \\
\le &\ 2\, A_{S-T}[\left|\psi_{MT}(\tau;t') - \psi_T(\tau;t')\right|].
\end{aligned}$$

Taking the mathematical expectation of the square of each side of this inequality yields

$$E\left|\Phi_M(nw_{S-T};t') - \Phi(nw_{S-T};t')\right|^2 \le 4\, E\{A^2_{S-T}[\left|\psi_{MT}(\tau;t') - \psi_T(\tau;t)\right|]\}$$

By Schwarz' inequality

$$A^2_{S-T}[\left|\psi_{MT}(\tau;t') - \psi_T(\tau;t')\right|] \le A_{S-T}[\left|\psi_{MT}(\tau;t') - \psi_T(\tau;t')\right|^2];$$

and thus we conclude from (5.10) that

$$\begin{aligned}
\lim_{M\to\infty} E\left|\Phi_M(nw_{S-T};t') - \Phi(nw_{S-T};t')\right|^2 &\le \lim_{M\to\infty} \{4\, A_{S-T}[E\left|\psi_{MT}(\tau;t') - \psi_T(\tau;t')\right|^2]\} \\
&\le 4\, A_{S-T}[\lim_{M\to\infty} E\left|\psi_{MT}(\tau;t') - \psi_T(\tau;t')\right|^2] = 0.
\end{aligned}$$

Therefore $P \lim\limits_{M\to\infty} \Phi_M(nw_{S-T};t') = \Phi(nw_{S-T};t')$ and $\Phi_M(nw_{S-T};t')$ is a consistent estimate of the spectral representation $\Phi(nw_{S-T};t')$.

7. Conclusions and Remarks

A set of functionals and their consistent estimators have been developed for the spectral analysis of continuous non-stationary random processes. The methodology is suitable for analysis of discrete parameter non-stationary processes by using appropriate summations in lieu of integrations. The methods were developed to analyze modulation frequencies in radar returns from space vehicles re-entering the earth's atmosphere. Good agreement is obtained between the spectral content of the radar returns through the use of these methods with data from other sensors. The method is most useful in determining spectral information from those processes whose frequencies change with time.

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